Left Zero Divisor Graphs of Totally Ordered Rings

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Abstract—*In this paper we consider prime graph of R (denoted by $PG(R)$) of an associative ring R (introduced by Satyanarayana, Syam Prasad and Nagaraju [6]). This short paper is divided into two Sections. Section-1 is devoted for preliminary definitions. In section-2, we constructed Left zero divisor graph of R (denoted by LZDG(R)) where R = the set of all 2×2 matrices over the ring $\mathbf{Z}_2$ of integers modulo 2.*

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I. INTRODUCTION

Let G = (V, E) be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge e_k is identified as an unordered pair of vertices {v_i, v_j}, where v_i, v_j are called end points of e_k. The edge e_k is also denoted by either v_iv_j or $\overline{v_iv_j}$. We also write $G(V,E)$ for the graph. Vertex set and edge set of G are also denoted by $V(G)$ and $E(G)$ respectively. An edge associated with a vertex pair {v_i, v_i} is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and δ(v) denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loop or parallel edges is called a simple graph. We consider simple graphs only. For an associative ring R, prime graph of R (denoted by $PG(R)$) was introduced in Satyanarayana, Syam Prasad and Nagaraju [6]. In this paper the concepts: left zero divisor graph was defined and compared with prime the graph.

1.1 Definitions: (i) A graph G (V, E) is said to be a star graph if there exists a fixed vertex v such that E = {vu / u ∈ V and u ≠ v}. A star graph is said to be an n-star graph if the number of vertices of the graph is n.

(ii) (Satyanarayana, Syam Prasad and Nagaraju [6]) Let R be an associative ring. A graph G (V, E) is said to be a ***prime graph*** of R (denoted by PG(R)) if V = R and E = { $\overline{xy}$ / xRy = 0 or yRx = 0, and x ≠ y}.

(iii) In a graph G, a subset S of V(G) is said to be a ***dominating set*** if every vertex not in S has a neighbour in S. The ***domination number***, denoted by γ(G) is defined as min {|S| / S is a dominating set in G}.

(iv) Let $\{1,2,3,\ldots,n\}$ be n objects to be permuted. For two permutations $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, we say that $a_1, a_2, \ldots, a_n$ comes before $b_1, b_2, \ldots, b_n$ in the **lexicographic order,** if some $1 \le m < n$, $a_1 = b_1$, $a_2 = b_2, \ldots, a_{m-1} = b_{m-1}$, and $a_m < b_m$.

(v) In a connected graph, a closed walk running through every vertex of G exactly once (except the starting vertex at which the walk terminates) is called as **Hamiltonian circuit.** A graph containing a Hamiltonian circuit is called as **Hamiltonian graph.**

(vi) Definition**:** A graph $G = (V, E)$ is said to be the zero divisor graph of R if $V = R$ and E = $\left\{\overline{xy}/x\neq y,\ x,y\in R,\ x\neq0\neq y,\ xy=0\right\} \cup \left\{\overline{x0}\ /\ 0\neq x\in R\right\}$ where $\overline{xy}$ denotes an edge between $x, y \in V$.

1.2 Note: (i) This definition 'zero divisor graph' is same as that of Beck [1988] in case of commutative rings.

1.3 Theorem: (Th. 13.8, page 361, [3]) A given connected graph G is an Eulerian graph if and only if all the vertices of G are of even degree.

For other fundamental concepts we refer [2], [3], [4] or [5]

In this paper, we use the following notation.

In this paper we consider totally ordered finite rings, the elements of a ring R with $|R| = n+1$ were totally ordered. Without loss of generality, we assume that $R = \{v_0, v_1, \ldots, v_n\}$ and the elements of R were totally ordered with $v_i < v_j$ if $i < j$.

II. LEFT ZERO DIVISOR GRAPHS

2.1 Definition: We define the left zero divisor graph of R (denoted by LZDG(R)), as follows;
$V(LZDG(R)) = R = \{v_i / 0 \le i \le n\}$, and $E(LZDG(R)) = \{\overline{v_i v_j} / v_i v_j = 0 \text{ with } i < j\}$.

2.2 Note: In case of commutative rings, LZDG(R) = ZDG(R).

2.3 Notation: Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let us identify it with the 4-tuple (a,b,c,d) with $a,b,c,d \in \mathbb{Z}_2$.

Lexicographic order on these 4–tuples is as follows:

$$(0,0,0,0) \le (0,0,0,1) \le (0,0,1,0) \le (0,0,1,1) \le (0,1,0,0) \le (0,1,0,1) \le (0,1,1,0) \le (0,1,1,1)$$
$$\le (1,0,0,0) \le (1,0,0,1) \le (1,0,1,0) \le (1,0,1,1) \le (1,1,0,0) \le (1,1,0,1) \le (1,1,1,0) \le (1,1,1,1)$$

Let us carry this lexicographic order to the 2×2 matrices, Then we have the following order:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \le \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \le \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \le \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \le \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \le \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Write $v_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v_5 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$
$v_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $v_7 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $v_8 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $v_9 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $v_{10} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $v_{11} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $v_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,
$v_{13} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $v_{14} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $v_{15} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then
$R = \{v_i / 0 \le i \le 15\}$ is an ordered ring with 16 elements and $v_0 \le v_1 \le \ldots \le v_{15}$ (In other words $v_i < v_j$ if $i < j$ for $0 \le i, j \le 15$,

2.4 Construction: In the following, we construct left zero divisor graph LZDG(R) of this ring R:
$V(LZDG(R)) = R = \{v_i / 0 \le i \le 15\}$.

Since $v_0 v_i = 0$ for all $1 \le i \le 15$ we have that $\overline{v_0 v_i} \in E(LZDG(R))$ for $1 \le i \le 15$.

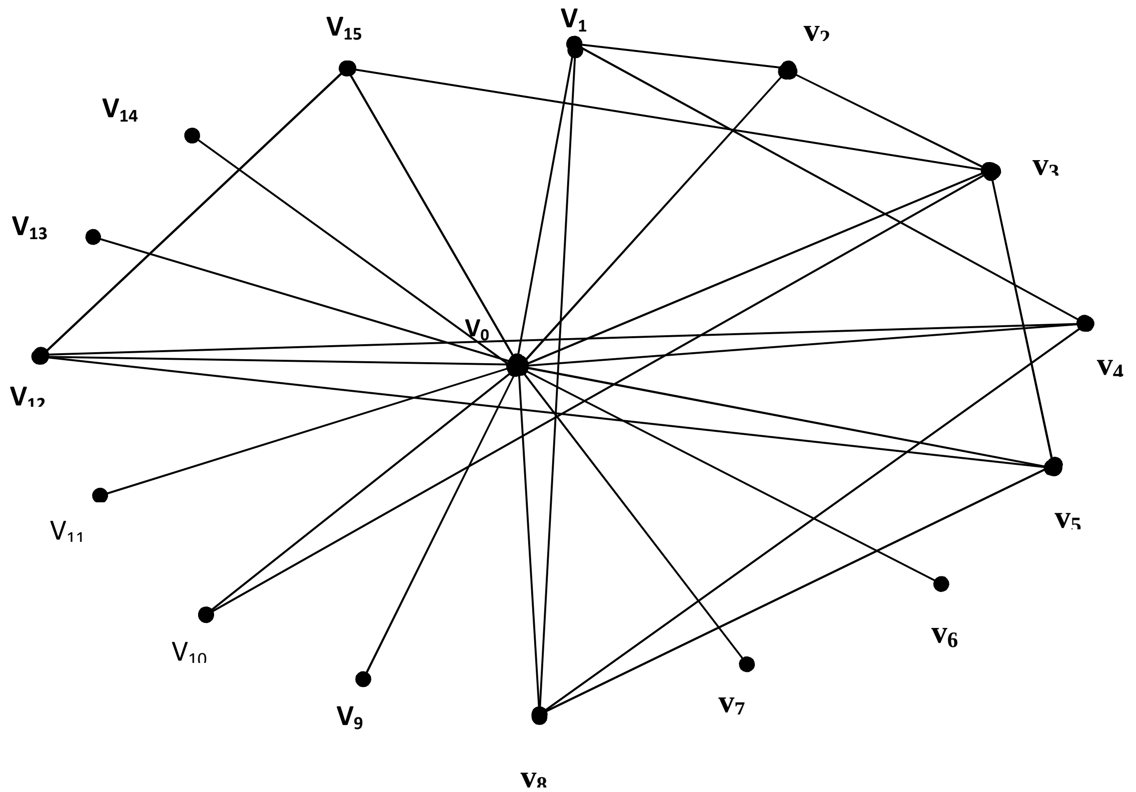
Also $v_1 v_2 = v_1 v_4 = v_1 v_8 = 0$ and $v_i v_j \ne 0$ for $i \notin \{0,2,4,8\}$,

$v_2 v_3 = v_3 v_5 = v_3 v_{10} = v_3 v_{15} = v_4 v_8 = v_4 v_{12} = v_5 v_8 = v_5 v_{12} = v_{12} v_{15} = 0$.

Therefore, $E(LZDG(R)) = \left\{ \begin{matrix} \overline{v_0v_1}, \overline{v_0v_2}, \overline{v_0v_3}, \overline{v_0v_4}, \overline{v_0v_5}, \overline{v_0v_6}, \overline{v_0v_7}, \overline{v_0v_8}, \overline{v_0v_9}, \overline{v_0v_{10}}, \overline{v_0v_{11}}, \overline{v_0v_{12}}, \overline{v_0v_{13}}, \overline{v_0v_{14}}, \overline{v_0v_{15}}, \\ \overline{v_1v_2}, \overline{v_1v_4}, \overline{v_1v_8}, \overline{v_2v_3}, \overline{v_3v_5}, \overline{v_3v_{10}}, \overline{v_3v_{15}}, \overline{v_4v_8}, \overline{v_4v_{12}}, \overline{v_5v_8}, \overline{v_5v_{12}}, \overline{v_{12}v_{15}} \end{matrix} \right\}$

.

The graph LZDG(R) is given in Figure 2.4.



2.5 Note: (i) (R, E_1) where $E_1 = \{\overline{v_j v_0} / 1 \le i \le 15\}$ is a 16 – star graph which is also a subgraph of LZDG(R).

(ii) The Domination number of LZDG(R) is 1.

(iii) Since $\overline{v_0 v_{15}}, \overline{v_{15} v_{12}}, \overline{v_{12} v_0}$ forms a triangle, we conclude that the graph cannot be a bipartite graph

(iv) LZDG(R) is not an Eulerian graph (by using the Th. 13.8, p 361 of [4]).

(v) Since LZDG(R) contains pendent vertices, it contains no Hamiltonian circuit.

(vi) In general $E(\text{LZD(R)}) \not\subset E((\text{PG(R)}))$. To see this, Consider the construction 2.4.

Since $0 \ne v_1 v_3 v_4 \in v_1 R v_4$ and $0 \ne v_4 v_1 v_1 \in v_4 R v_1$, we conclude that there is no edge between v_1 and v_4 in PG(R).

But $\overline{v_1 v_4} \in E(LZDG(R))$ (Refer: construction 2.4).

So $\overline{v_1 v_4} \in E(LZDG(R)) \setminus E(PG(R))$. Hence $E(\text{LZD(R)}) \not\subset E((\text{PG(R)}))$.

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